

Complex Coordinate Approaches with Applications to Perfectly Matched and Double Negative Layers

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Abstract—This paper presents a review of complex coordinate approaches used in electromagnetics, which are based on extending the real coordinate space to a complex coordinate space. After a general discussion of methods used in the literature, two special problems are discussed in detail: non-Maxwellian perfectly matched layer (PML) designs in the context of finite element method, and perfectly matched double negative (DNG) layer design. These are supported by some numerical simulations.

Index Terms—Complex coordinate, complex space, complex transformation, complex stretching, complex point-source, complex beam, complex ray, perfectly matched layer, double negative layer, metamaterial, transformation electromagnetics, transformation optics, anisotropic medium, transformation medium, electromagnetic scattering and radiation, computational electromagnetics, finite element method.

I. INTRODUCTION

Students or new beginners in engineering or physics usually come up with questions like: *Are complex numbers related to physical reality? How are they used in real life applications?* These are indeed deep questions with probably no trivial answers. It is usually believed that real numbers have physical reality because they are attributed to physical measurable quantities, but complex numbers have auxiliary or artificial nature in physics and can best be described by formal mathematics. However, a number (either real or complex) has a physical meaning if it can be linked to a theory in physics or engineering. Complex numbers, just like real numbers, are elements of mathematics, and can be used to model physical phenomena. They have a wide range of applications, such as signal representations, Fourier analysis, phasor analysis, etc. in engineering and physics.

The purpose of this review paper is to present a unifying point of view that discusses the use of *complex coordinates* or *complex space* in electromagnetics. The key role of a complex space point is to introduce loss (dissipation) or gain into the problem of interest. This property can be used directly, for example to model losses in a medium, or can be utilized to create new functionalities with different purposes.

The first use of the concepts of *complex point-source* (i.e., point source located at a complex space point) and *complex rays* can be traced back to Keller's 1958 paper [1], followed by similar approaches to solve various scattering and diffraction problems in the literature [2]-[17]. It was shown that the concept of complex point-source can be used to generate Gaussian beams [3], [4]. The Green's function expression e^{-jkr}/r evaluated at a point P, where r is the distance between the source located at a complex point and the observation point P, represents the field of a Gaussian beam. Beams that are created by complex point-sources are exact directional fields determined by an analytical extension of the Green's function into the complex space. In other words, the main mission of shifting points to complex space is to enable point sources to radiate in specific angles by providing decay in other angles. The field patterns of complex-point sources are similar to real-point sources except for the exponential angular decay. Note that the *decay* feature associated with complex coordinates is utilized to model sources with directionality. But, this decay feature was also used to solve scattering problems in lossy media by using the concept of complex ray tracing [5], [6]. Apart from these studies, Boag and Mittra showed that the matrix size of the method of moments solution of electromagnetic scattering problems can be reduced by representing the scattered field in terms of a series of beams produced by multipole sources located in complex space [16], [17]. The directional nature of the fields radiated by multipole sources, located in complex space, ensures that the coupling between distant parts of an object is low, and hence, matrix elements corresponding to low coupling can be eliminated.

Perhaps, the most widely used application of complex coordinate transformation is the design of perfectly matched layer (PML), which is used as a numerical absorber in mesh truncation of finite methods (such as finite element method (FEM), and finite difference time domain (FDTD) method) for solving open-region radiation and scattering problems. Such a complex coordinate transformation is known as *complex coordinate stretching* in PML nomenclature. With the help of complex coordinates in the PML region, outgoing waves are replaced by *exponentially decaying waves* so that outgoing waves are absorbed without any reflection irrespective of their frequency and angle of incidence. The main superiority of the PML approach over absorbing

boundary conditions is that PMLs have the ability to minimize white space by locating it to the close proximity to the surface of the volume of scatterers or sources. In fact, the PML concept was implemented by means of different approaches and interpretations in the literature. It was first introduced by Berenger [18] in the context of the FDTD method, by using a split-field formulation of Maxwell's equations in Cartesian coordinates. This approach yields non-Maxwellian fields within PML. Afterwards, Sacks et al. realized the Maxwellian PML in Cartesian coordinates as an anisotropic layer with permittivity and permeability tensors in the context of FEM [19]. Gedney used the concept of anisotropic PML medium in FDTD [20]. These anisotropic PML approaches were originally introduced in Cartesian coordinates. Kuzuoglu and Mittra extended this concept to cylindrical and spherical coordinates [21], and to the design of conformal PMLs using a local curvilinear coordinate system [22], in the context of the FEM. It was shown by Chew et. al. that the above approaches are equivalent to a more general approach, called *complex coordinate stretching* [23], [24]. This non-Maxwellian approach is based on the analytical continuation of the field variables to complex space. Teixeira and Chew showed that anisotropic PML can be realized by applying a complex coordinate stretching through the mapping of the non-Maxwellian fields obtained during the complex stretching to a set of Maxwellian fields in an anisotropic medium representing the PML [25]-[27].

The above-mentioned PML implementations, especially in the context of FEM, are based on the design of an anisotropic medium having suitably defined constitutive parameters. In addition, all of them were proposed in rectangular, circular or spherical domains that do not have arbitrary curvature discontinuities. Although the concept of local coordinates was used for designing conformal PMLs, its implementation is difficult especially for arbitrary geometries. Ozgun and Kuzuoglu proposed an approach, called *locally-conformal PML*, based on complex coordinate stretching to design conformal PMLs over challenging geometries, especially having some intersection regions or abrupt changes in curvature, in an easier and flexible manner [28]-[30]. Such conformal PML domains are vital to decrease white space, thereby to decrease computational load. This method does not employ the concept of anisotropic medium, and only replaces the real coordinates with their complex counterparts obtained by a complex coordinate transformation. Since the transformation is locally defined, it has no explicit dependence on the differential geometric characteristics of the PML-free space interface, and hence it becomes possible to model arbitrary PML geometries. Another method, called *multi-center PML method*, was also proposed for the design of conformal PMLs [31], [30]. Both methods will be summarized in Sec. II. Due to its simple and flexible nature, the locally-conformal PML method was also employed to divide a computational domain into a number of subdomains in domain decomposition methods to eliminate artificial reflections arising from the abrupt division of the domain [32]-[35]. It was also used to truncate infinitely-long conducting structures in a finite-sized domain to be able to achieve FEM modeling of high frequency diffraction problems [36].

In recent years, the coordinate transformation approach was also employed in designing metamaterial structures. In [37], Kuzuoglu extended the concept of a double negative (DNG) medium (an isotropic medium with negative permittivity and permeability), which was proposed by Veselago [38], to a perfectly matched DNG medium by means of *complex* coordinate transformations. He demonstrated the anisotropic generalizations of the Veselago medium. This approach will be explained in Sec. III. The concept of coordinate transformations was also combined with the *transformation optics* (TO) (or *transformation electromagnetics* (TEM)) approach. The TO approach is an intuitive and systematic approach to the design of metamaterials (or application-oriented transformation media), and it is based on "*real*" coordinate transformations to manipulate the behavior of electromagnetic waves in a desired manner. It uses the form invariance property of Maxwell's equations under coordinate transformations [39]. When the spatial domain of a medium is modified by employing a coordinate transformation, this medium equivalently turns into an anisotropic medium in which Maxwell's equations have the same mathematical form. The constitutive parameters of the anisotropic medium are obtained by the Jacobian of the coordinate transformation. The central concept of TO can be traced back to Bateman's study in 1910 [40]. It was also discussed in the context of finite methods in [41], [42]. Over the last decade, the TO approach—which is based on real coordinates—was employed to the design of various optical and electromagnetic structures, such as invisibility cloak [43]-[45], electromagnetic reshaper [46]-[49], waveguide miniaturization [50]-[52], lenses [47], [53], [54], concentrators [55], [56], holes [57], antennas [58], [59], etc. Other than these applications that aim to design devices with new capabilities, the TO approach was also used to design computational materials (called *software metamaterials*) to overcome some numerical difficulties that arise in computational electromagnetics problems, such as modeling large-scale or multi-scale electromagnetic scattering problems [60], [61]; modeling curved geometries that do not conform to a Cartesian grid [62], [63], and modeling stochastic electromagnetic problems modeling uncertainties [64]-[67]. In recent years, the TO approach was extended, via complex extension of the spatial coordinates, to achieve field-amplitude control [68], to create single-negative [69] and $\mathcal{PT}$ -symmetric [70] transformation media, to handle non-Hermitian optical problems [71], and to design lenses [72]. Also, the use of complex coordinates was investigated to understand the effect of inhomogeneous media on wave propagation [73]. In these applications, the transformation medium involves both dissipation and gain due to material parameters that are complex-valued functions of position.

The organization is as follows: In Sec. II, locally-conformal and multi-center PML approaches are summarized. In Sec. III, the design of perfectly matched DNG medium is presented. In Sec. IV, some conclusions are drawn. Throughout the paper, the suppressed time dependence of the form $\exp(j\omega t)$ is assumed.

II. PERFECTLY MATCHED LAYER (PML) DESIGN

This section presents two PML approaches based on complex coordinate approaches: *locally-conformal PML* and *multi-center PML*. These are non-Maxwellian PML methods that do not require any artificial anisotropic medium and coordinate system. These methods use specially- and locally-defined complex coordinate stretching, and hence, make possible the easy design of conformal PMLs having challenging geometries.

Before the details of these special methods, it is useful to discuss the principal idea in a *classical PML approach* based on complex coordinates. Consider first the simplest case as shown in Fig. 1, where a TM_z plane wave (i.e., $\mathbf{E} = \hat{a}_z E_z(x, y)$) with arbitrary frequency and direction of propagation is incident to an interface (Γ) that separates two half spaces ($x > 0$ and $x < 0$). The PML and free space (FS) regions are defined as $\Omega_{\text{PML}} = \{(x, y) | x > 0\}$ and $\Omega_{\text{FS}} = \{(x, y) | x < 0\}$, respectively. In Ω_{FS} , the plane wave is given by

$$E_z(x, y) = e^{-jk_x x} e^{-jk_y y}, \quad (1)$$

where $k_x = k \cos \theta$, $k_y = k \sin \theta$, k is the wave number and θ is the angle between the direction of propagation and the x -axis. The purpose of the PML approach is to transmit the plane wave into the PML region Ω_{PML} without any reflection, and to attenuate it in the $+x$ direction. Therefore, it is stipulated that the plane wave in Ω_{PML} is given by

$$E_z(x, y) = f(x) e^{-jk_x x} e^{-jk_y y}, \quad (2)$$

where $f(x)$ is a function satisfying the following two conditions:

- $f(0) = 1$,
- $f(x)$ decreases monotonically for $x > 0$.

For example, $f(x)$ can be chosen as

$$f(x) = e^{-\alpha x \cos \theta}, \quad (3)$$

where $\alpha > 0$ is a constant. Note that the angular dependence of the decay characteristic is needed to satisfy reciprocity in PML. Now, define a parameter a as

$$a = 1 + \frac{\alpha}{jk}. \quad (4)$$

By substituting (4) into (3), and then (3) into (2), we get

$$E_z(x, y) = e^{-jk_x a x} e^{-jk_y y}. \quad (5)$$

At this point, a *complex* coordinate $\tilde{x}$ can be defined as

$$\tilde{x} = ax, \quad (6)$$

or equivalently,

$$\tilde{x} = x + \frac{\alpha}{jk} x. \quad (7)$$

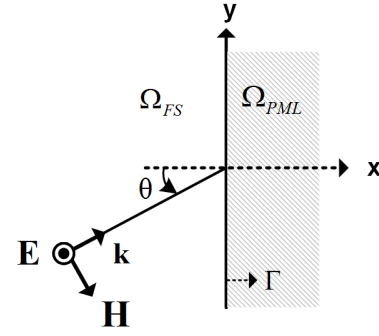


Fig. 1. A TM_z plane wave is incident on a PML-free space interface.

Note that, the real coordinate is transformed to a complex coordinate by just adding an imaginary part to the real coordinate x . With this complex coordinate, (5) becomes

$$E_z(\tilde{x}, y) = e^{-jk_x \tilde{x}} e^{-jk_y y}. \quad (8)$$

This equation satisfies Helmholtz equation

$$\frac{\partial^2 E_z}{\partial \tilde{x}^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = 0. \quad (9)$$

everywhere on $\mathbb{R}^2$, provided that

$$\tilde{x} = \begin{cases} x, & x < 0 \\ x + \frac{\alpha}{jk} x, & x > 0 \end{cases}. \quad (10)$$

One attractive consequence of this interpretation is that it allows to solve the standard Helmholtz equation with a standard finite element method except that *the real coordinates are simply replaced by complex counterparts*.

The above formulation is based on the fact that the PML region is a half space. Indeed, the PML region must be a bounded region enclosing all sources and objects. The above formulation can easily be extended to a rectangular layer in 2-D Cartesian coordinates, a part of which is shown in Fig. 2(a). The inner and outer boundaries are represented by Γ_i and Γ_o , respectively. A successful PML design ensures that the waves incident to Γ_i from the free space domain Ω_{FS} are transmitted into Ω_{PML} without any reflection in order not to affect the fields within the free space region, and they decay as they traverse the PML region toward the outer boundary. Since the magnitude of the field reduces to a negligible value at the outer boundary, this boundary can be modeled as a perfect electric or magnetic conductor. In Fig. 2(a), the PML region is represented as a union of three regions. In regions I and II, the waves must be attenuated along the x - and y -directions, respectively. However, in the corner region III, the waves must decay in both directions. Hence, both coordinates are transformed to complex coordinates to attenuate the waves in an oblique direction that is a combination of the x - and y -directions. The complex coordinate transformation can be defined as follows:

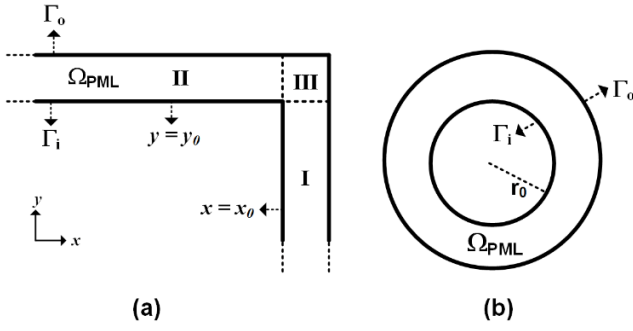


Fig. 2. Classical PML methods: (a) Cartesian, (b) Cylindrical.

$$\text{Region I: } \tilde{x} = x + \frac{\alpha_1}{jk}(x - x_0) \text{ and } \tilde{y} = y \quad (11.a)$$

$$\text{Region II: } \tilde{x} = x \text{ and } \tilde{y} = y + \frac{\alpha_2}{jk}(y - y_0) \quad (11.b)$$

$$\text{Region III: } \tilde{x} = x + \frac{\alpha_1}{jk}(x - x_0) \text{ and } \tilde{y} = y + \frac{\alpha_2}{jk}(y - y_0) \quad (11.c)$$

Note that α_1 and α_2 might be chosen as identical if the rate of decay in both directions is desired to be the same. With this transformation, the plane wave expression is given by

$$E_z(x, y) = \begin{cases} e^{-jk_x a x} e^{-jk_y y}, & \text{in region I} \\ e^{-jk_x x} e^{-jk_y b y}, & \text{in region II} \\ e^{-jk_x a x} e^{-jk_y b y}, & \text{in region III} \end{cases} \quad (12)$$

where $a = 1 + (\alpha_1 / jk)$ and $b = 1 + (\alpha_2 / jk)$.

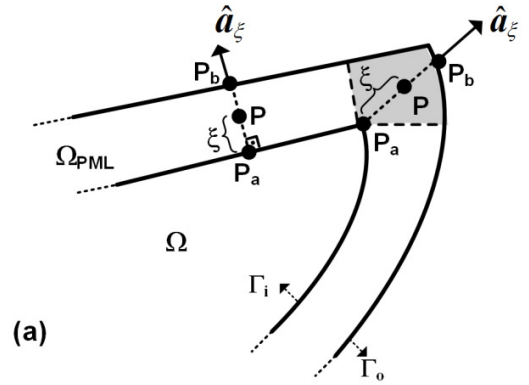
A similar approach can be developed if the PML region surrounds a circular spatial domain as shown in Fig. 2(b). Using the cylindrical coordinates, the waves are attenuated in the radial direction by defining the transformation as follows:

$$\tilde{r} = r + \frac{\alpha}{jk}(r - r_0). \quad (13)$$

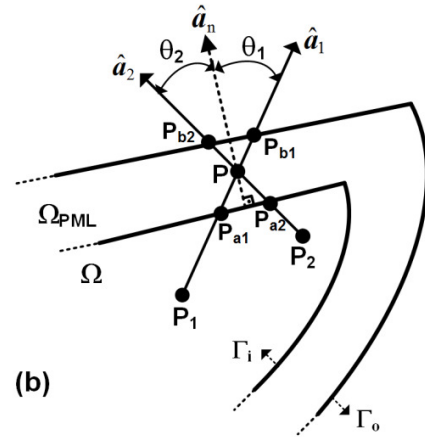
Note that these classical PML approaches *fail* if the PML domain is of arbitrary shape, because inner and outer PML surfaces must be defined on “constant coordinate surfaces”. The locally-conformal and multi-center PML methods (described below) overcome this difficulty, and the inner and outer PML surfaces need not be identical, providing a great flexibility in designing arbitrarily-shaped PML regions.

A. Locally-conformal PML approach

The method is illustrated in Fig. 3(a), which shows the spatial region occupied by a convex PML region (Ω_{PML}). The PML can be designed as conformal to an arbitrary volume (Ω) containing sources and objects. The PML is realized by mapping each point P in $\Omega_{\text{PML}} \in \mathbb{R}^3$ to $\tilde{P}$ in complex PML region $\tilde{\Omega}_{\text{PML}} \in \mathbb{C}^3$, through the complex coordinate transformation as follows:



(a)



(b)

Fig. 3. Conformal PML methods based on coordinate stretching: (a) Locally-conformal PML, (b) Multi-center PML.

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{1}{jk} f(\xi) \hat{\mathbf{a}}_\xi, \quad (14)$$

where $\mathbf{r} \in \mathbb{R}^3$ and $\tilde{\mathbf{r}} \in \mathbb{C}^3$ are the position vectors of the points P in real space and $\tilde{P}$ in complex space, respectively; k is the wavenumber; and ξ is the parameter defined by

$$\xi = \|\mathbf{r} - \mathbf{r}_a\|, \quad (15)$$

where $\mathbf{r}_a \in \Gamma_i$ is the position vector of the point P_a located on the inner PML boundary Γ_i , which is obtained by solving the following minimization problem:

$$\min_{\mathbf{r}_a \in \Gamma_i} \|\mathbf{r} - \mathbf{r}_a\|. \quad (16)$$

This implies that ξ is the shortest distance between the inner boundary and the corresponding PML point. Eq. (16) results in a unique $\mathbf{r}_a$ because Γ_i is the boundary of the convex set. Moreover, $\hat{\mathbf{a}}_\xi$ is the unit vector defined by

$$\hat{\mathbf{a}}_\xi = (\mathbf{r} - \mathbf{r}_a) / \xi, \quad (17)$$

and $f(\xi)$ is a monotonically increasing function of ξ as follows:

$$f(\xi) = \frac{\alpha \xi^m}{m \|\mathbf{r}_b - \mathbf{r}_a\|^{m-1}}, \quad (18)$$

where α is a positive parameter (practically, $5k \leq \alpha \leq 15k$ if PML thickness is between $\lambda/4$ and $\lambda/2$ where λ is the wavelength), m is a positive integer (typically 2 or 3) related to the decay rate of the magnitude of the field inside PML, and $\mathbf{r}_b$ is the position vector of the point P_b which is the intersection of the line passing through $\mathbf{r}$ and $\mathbf{r}_b$. Note that $\|\mathbf{r}_b - \mathbf{r}_a\|$ is the local PML thickness for the corresponding PML point.

B. Multi-center PML approach

This method (shown in Fig. 3(b)) is basically a generalization of the locally-conformal PML method. It is implemented by choosing a finite number of center points (such as P_1 and P_2) inside the volume Ω . The unit vectors ($\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$) emanates from these centers in the direction of the PML point, P. The unit vector $\hat{\mathbf{a}}_n$, shown by the dotted line, is equivalent to $\hat{\mathbf{a}}_\xi$ in (17). Furthermore, θ_1 and θ_2 are the angles between $\hat{\mathbf{a}}_n$ and the unit vectors $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, respectively. The complex coordinate transformation, which maps each point P inside the PML region to $\tilde{\mathbf{P}}$ in complex PML region, is defined as follows:

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{\alpha}{jk} \sum_{i=1}^N w_i \frac{\|\mathbf{r} - \mathbf{r}_{a,i}\|^m}{m \|\mathbf{r}_{b,i} - \mathbf{r}_{a,i}\|^{m-1}} \hat{\mathbf{a}}_i, \quad (19)$$

where α and m are the same as before, N is the number of the centers, and w_i is the real weight value assigned to each center P_i . The sum of all weights should add up to 1, and each weight w_i is inversely proportional to the angle θ_i , which is less than or equal to 45° . The value 45° is the threshold value in assigning the weights, and has been determined empirically by means of systematic numerical experiments. The weight selection scheme assigns the highest priority to the center whose unit vector $\hat{\mathbf{a}}_i$ makes the smallest angle with $\hat{\mathbf{a}}_n$. To achieve smooth decay inside the PML region, the centers should be chosen in such a way that each PML point should have at least one center, whose angle θ_i is less than or equal to 45° . In some smooth geometries (such as a spherical or cubical shell), a single center-of-mass point can provide reliable results. The center selection scheme is a straightforward task depending on the geometry of the PML region.

Both locally-conformal and multi-center PML transformations can be implemented by just simply replacing the real-valued node coordinates inside PML by their complex-valued counterparts calculated via transformations. Another way is to model the PML region as an anisotropic medium whose constitutive parameters are obtained by the form invariance property of Maxwell's equations under coordinate transformations. This approach will be summarized in Sec. II.C for the sake of completeness.

Let us now discuss how the transformations achieve field decay inside the PML region. Consider an outgoing wave in the neighborhood of an arbitrary point P_a , which can be locally represented as a superposition of plane waves. A typical representative is given as follows:

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{a}}_p e^{-jk\hat{\mathbf{a}}_k \cdot \mathbf{r}}, \quad (20)$$

where $\hat{\mathbf{a}}_k$ is the unit vector along the direction of incidence, and $\hat{\mathbf{a}}_p$ is the unit vector denoting the polarization. Let $\tilde{\mathbf{E}}(\tilde{\mathbf{r}})$ be the transformed version of $\mathbf{E}(\mathbf{r})$ to complex space given by the expression

$$\tilde{\mathbf{E}}(\tilde{\mathbf{r}}) = \hat{\mathbf{a}}_p e^{-jk\hat{\mathbf{a}}_k \cdot \tilde{\mathbf{r}}}. \quad (21)$$

By using the coordinate transformations, the field inside the PML region becomes

$$\tilde{\mathbf{E}}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) g(\mathbf{r}), \quad (22)$$

where

$$g(\mathbf{r}) = \exp\left[-f(\xi) \hat{\mathbf{a}}_k \cdot \hat{\mathbf{a}}_\xi\right], \quad (\text{local-conformal}) \quad (23a)$$

$$g(\mathbf{r}) = \exp\left[-\sum_{i=1}^N w_i \frac{\alpha \|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_{a,i}\|^m}{m \|\tilde{\mathbf{r}}_{b,i} - \tilde{\mathbf{r}}_{a,i}\|^{m-1}} \hat{\mathbf{a}}_i \cdot \hat{\mathbf{a}}_k\right], \quad (\text{multi-center}) \quad (23b)$$

This function satisfies the following two conditions:

- (i) $g(\mathbf{r})|_{\text{on } \Gamma_i} = 1$. It assures the continuity of field at the interface to get rid of numerical reflections.
- (ii) It is a monotonically decreasing function away from the PML-free space interface.

Consequently, the coordinate transformations satisfy the following three conditions that are essential for a successful PML design:

- (i) the outgoing wave in the neighborhood of Γ_i is transmitted into Ω_{PML} without any reflection,
- (ii) the transmitted wave decays monotonically within Ω_{PML} along the direction of the unit vector and away from the PML-free space interface,
- (iii) the magnitude of the transmitted wave is negligible on Γ_o .

The performances of the PML methods are tested by FEM simulations considering the TM_z radiation of a line-source inside an infinitely-long cylindrical region of hexagonal shape. The position of the line-source is assumed to be a "random variable" uniformly distributed inside the region, based on the assumption that the performance of the PML may depend on the position of the source. For this purpose, we utilize the Monte Carlo simulation technique, which is a method based on the use of repeated random simulations to extract the statistical parameters of stochastic problems, to show that the PML methods provide reliable and robust results irrespective of the source position. We randomly determine 2000 different source positions, and run the FEM program 2000 times. For each run, we calculate the mean-square error by comparing the electric field calculated by the FEM, which is truncated by either locally-conformal or

multi-center PML, with the analytical solution. In Fig. 4, we plot the error scatter plot that shows the error value at each source position, as well as the error histogram. We also show some statistical data (i.e., mean, variance, etc.) on the plots.

C. Maxwellian PML implementation via form invariance property of Maxwell's equations

The non-Maxwellian PML methods described above are implemented by simply interchanging the real coordinates by complex-valued coordinates in a standard FEM code. They do not model the PML as an anisotropic medium. However, the approaches are indeed equivalent to anisotropic medium design through the use of form invariance property of Maxwell's equations. The medium in which the transformation is applied turns into an anisotropic medium with constitutive parameters obtained by the Jacobian of the transformation. Transformed fields in the anisotropic medium satisfy the original form of Maxwell's equations. This approach is briefly summarized below.

If the original medium before the transformation is an isotropic medium with parameters (ϵ, μ) , then the permittivity and permeability tensors of the anisotropic PML medium are determined by

$$\bar{\bar{\epsilon}}_{PML} = \epsilon \bar{\bar{\Lambda}} \quad \text{and} \quad \bar{\bar{\mu}}_{PML} = \mu \bar{\bar{\Lambda}}, \quad (24)$$

where

$$\bar{\bar{\Lambda}} = \delta_{\bar{\bar{J}}} (\bar{\bar{J}}^T \cdot \bar{\bar{J}})^{-1}, \quad (25)$$

and $\bar{\bar{J}} = \partial(\tilde{x}, \tilde{y}, \tilde{z}) / \partial(x, y, z)$. Here, $\bar{\bar{J}}$ is the Jacobian tensor and $\delta_{\bar{\bar{J}}}$ is its determinant. If the original medium is an anisotropic medium with parameters $(\bar{\bar{\epsilon}}, \bar{\bar{\mu}}')$, then the parameters of the anisotropic PML medium are obtained by

$$\bar{\bar{\epsilon}}_{PML} = \delta_{\bar{\bar{J}}} (\bar{\bar{J}}^{-1})^T \cdot \bar{\bar{\epsilon}}' \cdot (\bar{\bar{J}}^{-1}), \quad (26a)$$

$$\bar{\bar{\mu}}_{PML} = \delta_{\bar{\bar{J}}} (\bar{\bar{J}}^{-1})^T \cdot \bar{\bar{\mu}}' \cdot (\bar{\bar{J}}^{-1}). \quad (26b)$$

Assuming that the wave propagation is along the infinite z -axis (i.e., no z -variation), z -dependent off-diagonal terms of the Jacobian tensor become zero (i.e., $\partial\tilde{x}/\partial z = \partial\tilde{y}/\partial z = \partial\tilde{z}/\partial x = \partial\tilde{z}/\partial y = 0$), as well as the z -dependent diagonal term becomes unity $\partial\tilde{z}/\partial z = 1$.

Based on the form invariance property of Maxwell's equations, the electric field satisfies the following 3-D vector wave equation in transformed and original coordinates, respectively, as follows:

$$\tilde{\nabla} \times \tilde{\nabla} \times \mathbf{E}(\tilde{\mathbf{r}}) - k^2 \mathbf{E}(\tilde{\mathbf{r}}) = 0, \quad (27a)$$

$$\nabla \times [\bar{\bar{\Lambda}}^{-1} \cdot \nabla \times \tilde{\mathbf{E}}(\mathbf{r})] - k^2 \bar{\bar{\Lambda}} \cdot \tilde{\mathbf{E}}(\mathbf{r}) = 0, \quad (27b)$$

where $\tilde{\nabla} = [\bar{\bar{J}}^{-1}]^T \cdot \nabla$ is the del operator in transformed space, $\mathbf{E}(\tilde{\mathbf{r}})$ is the original field in transformed coordinates, and $\tilde{\mathbf{E}}(\mathbf{r})$ is the transformed field in original coordinates. These fields are interrelated through the following principle of field equivalence:

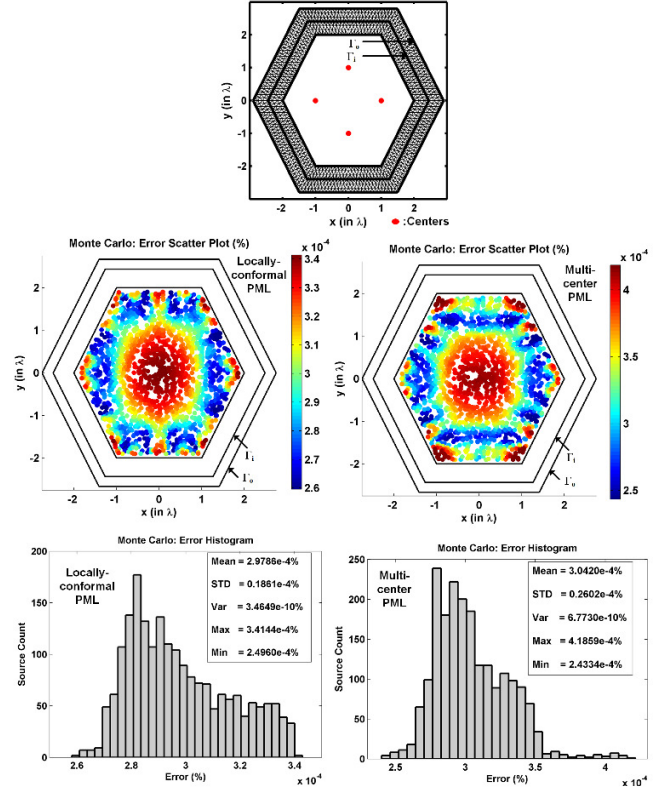


Fig. 4. Monte Carlo simulations to test the performance of the PML methods: (Top) Mesh structure, (Middle) Error scatter plots, (Bottom) Histogram plots.

$$\tilde{\mathbf{E}}(\mathbf{r}) = \bar{\bar{J}}^T \cdot \mathbf{E}(\tilde{\mathbf{r}}), \quad (28a)$$

$$\tilde{\mathbf{H}}(\mathbf{r}) = \bar{\bar{J}}^T \cdot \mathbf{H}(\tilde{\mathbf{r}}). \quad (28b)$$

For TM_z case, the vector wave equations in (27a) and (27b) reduce to scalar Helmholtz equations in transformed and original coordinates, as follows:

$$\tilde{\nabla}^2 E_z + k^2 E_z = 0, \quad (29a)$$

$$\nabla \cdot (\bar{\bar{\Lambda}}_t \nabla \tilde{E}_z) + k^2 \delta_t \Lambda_{33} \tilde{E}_z = 0, \quad (29b)$$

where $\bar{\bar{\Lambda}}_t$ is the 2×2 tensor corresponding to the transverse part of $\bar{\bar{\Lambda}}$, δ_t is its determinant, and Λ_{33} is the (3,3) component of $\bar{\bar{\Lambda}}$. Because of the special form of the Jacobian tensor in TM_z case, (28a) is expressed as $E_z(\tilde{x}, \tilde{y}) = \tilde{E}_z(x, y)$.

III. PERFECTLY MATCHED DOUBLE NEGATIVE LAYER (DNG) DESIGN

In a double negative (DNG) medium, which is also called the Veselago medium, the wave vector $\mathbf{k}$ and Poynting's vector $\mathbf{P}$ are anti-parallel, that causes negative refraction and reversal of Doppler shift and Cherenkov radiation. In this section, we model a perfectly matched DNG medium by using complex coordinate transformations, and show how to obtain anisotropic generalizations of the Veselago medium. It should be noted that although the coordinate

transformations employed in this section are eventually reduced to real-valued mappings, the ideas are very similar to those in the previous section.

Let us define a complex coordinate mapping $T: \mathcal{R}^3 \rightarrow \mathcal{C}^3$, which applies the transformation only in x -direction, as:

$$\tilde{\mathbf{r}} = (x, y, z) \rightarrow \tilde{\tilde{\mathbf{r}}} = T(\tilde{\mathbf{r}}) = (\tilde{\tilde{x}}, \tilde{y}, \tilde{z}), \quad (30)$$

where

$$\tilde{\tilde{x}} = \begin{cases} x & x < 0 \\ \int_0^x s_x(x') dx' & x > 0 \end{cases} \quad \text{and} \quad \tilde{y} = y \quad \text{and} \quad \tilde{z} = z \quad (31)$$

where the stretching parameter s_x is in the complex form $s_x(x') = s_{xr}(x') + js_{xi}(x')$. Hence, in the half space $x > 0$, the nabla operator $\tilde{\nabla}$ in the complex space and the tensor $\tilde{\tilde{\Lambda}}$ in (25), respectively, become

$$\tilde{\nabla} = \begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nabla \quad (32)$$

$$\tilde{\tilde{\Lambda}} = \begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix} \quad (33)$$

because

$$\frac{\partial}{\partial x} = s_x \frac{\partial}{\partial \tilde{\tilde{x}}} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \tilde{z}} \quad (34)$$

Now, let $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ be the transformed fields (i.e., mapped versions of $(\mathbf{E}, \mathbf{H})$ to $\mathcal{C}^3$) by restricting them to $\Gamma \subset \mathcal{C}^3$, which is defined as the manifold $\Gamma = \{(\tilde{\tilde{x}}, \tilde{y}, \tilde{z}) \in \mathcal{C}^3 | T(x, y, z) = (\tilde{\tilde{x}}, \tilde{y}, \tilde{z}), (x, y, z) \in \mathcal{R}^3\}$. Thus, the governing equations in the region $\Omega_+ = \{(x, y, z) \in \mathcal{R}^3 | x > 0\}$ become

$$\tilde{\nabla} \times \tilde{\mathbf{E}} = -j\omega\mu_0 \tilde{\mathbf{H}} \quad (35.a)$$

$$\tilde{\nabla} \times \tilde{\mathbf{H}} = j\omega\epsilon_0 \tilde{\mathbf{E}} \quad (35.b)$$

Hence, the field distribution $(\mathbf{E}^a, \mathbf{H}^a)$ inside the anisotropic medium occupying the region Ω_+ satisfy Maxwell's equations with constitutive parameters $\tilde{\tilde{\mu}} = \mu_0 \tilde{\tilde{\Lambda}}$ and $\tilde{\tilde{\epsilon}} = \epsilon_0 \tilde{\tilde{\Lambda}}$, and can be expressed as

$$\mathbf{E}^a = s_x \tilde{\tilde{E}}_x \hat{\mathbf{a}}_x + \tilde{\tilde{E}}_y \hat{\mathbf{a}}_y + \tilde{\tilde{E}}_z \hat{\mathbf{a}}_z \quad (36.a)$$

$$\mathbf{H}^a = s_x \tilde{\tilde{H}}_x \hat{\mathbf{a}}_x + \tilde{\tilde{H}}_y \hat{\mathbf{a}}_y + \tilde{\tilde{H}}_z \hat{\mathbf{a}}_z \quad (36.b)$$

The stretching variable s_x is in general a function of the space variable x . Let us assume that s_x is a complex constant given by

$$s_x = \alpha - j\sigma \quad (37)$$

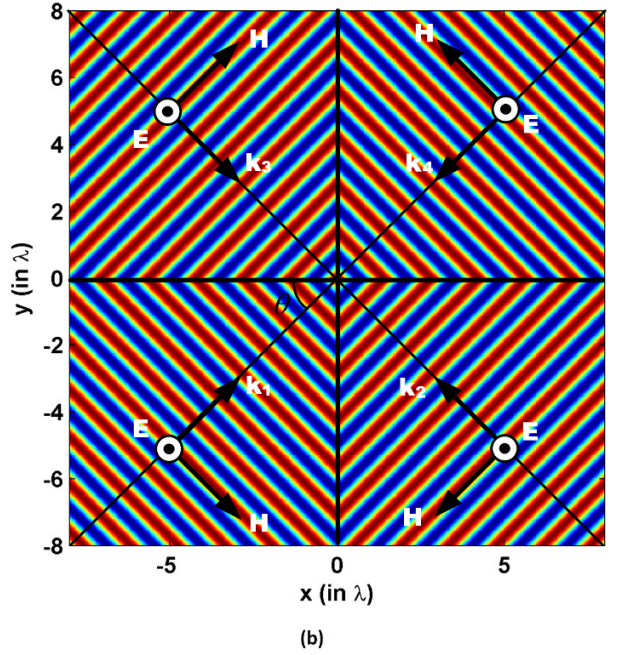
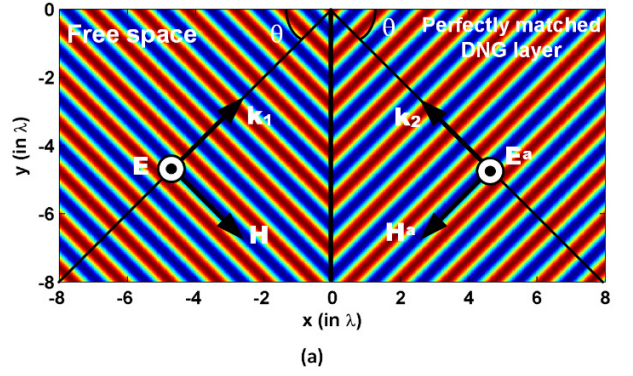


Fig. 5. Transmission of a plane wave into the perfectly matched DNG medium: (a) Coordinate stretching in x -direction, (b) Coordinate stretching in x - and y -directions (Overlapping perfectly matched DNG layers)

where α and $\sigma \geq 0$ are real constants. In the region Ω_+ , the space variable in the transformed space becomes

$$\tilde{\tilde{x}} = (\alpha - j\sigma)x \quad (38)$$

In the limit as $\sigma \rightarrow 0$, the anisotropic medium becomes isotropic. If $\alpha = 1$, the medium is free space ($\epsilon = \epsilon_0$ and $\mu = \mu_0$), whereas if $\alpha = -1$, it is a Veselago medium ($\epsilon = -\epsilon_0$ and $\mu = -\mu_0$).

Let us now illustrate how this transformation creates negative refraction in the Veselago medium with $\epsilon = -\epsilon_0$ and $\mu = -\mu_0$. We assume that a TM_z plane wave $(\mathbf{E}, \mathbf{H})$ exists in the half-space region $\Omega_- = \{(x, y, z) \in \mathcal{R}^3 | x < 0\}$. Defining the propagation vector as $\mathbf{k}_1 = k(\hat{\mathbf{a}}_x \cos \theta + \hat{\mathbf{a}}_y \sin \theta)$, the fields can be expressed as follows:

$$\mathbf{E} = \hat{\mathbf{a}}_z \exp[-jk(x \cos \theta + y \sin \theta)] \quad (39a)$$

$$\begin{aligned} \mathbf{H} &= \frac{1}{\eta_0} [\hat{\mathbf{a}}_x \sin \theta - \hat{\mathbf{a}}_y \cos \theta] \exp[-jk(x \cos \theta + y \sin \theta)] \\ &= \frac{1}{\omega \mu_0} (\mathbf{k}_1 \times \mathbf{E}) \end{aligned} \quad (39b)$$

where η_0 is the intrinsic impedance of free space. The triplet $(\mathbf{E}, \mathbf{H}, \mathbf{k}_1)$ is right-handed, as illustrated in Fig. 5(a). In the half-space Ω_+ , the analytic continuation of $\mathbf{E}$ and $\mathbf{H}$ to complex space becomes

$$\begin{aligned} \tilde{\mathbf{E}} &= \hat{\mathbf{a}}_z \exp[-jk(\tilde{x} \cos \theta + \tilde{y} \sin \theta)] \\ &= \hat{\mathbf{a}}_z \exp[-jk((-x) \cos \theta + y \sin \theta)] \end{aligned} \quad (40a)$$

$$\begin{aligned} \tilde{\mathbf{H}} &= \frac{1}{\eta_0} [\hat{\mathbf{a}}_x \sin \theta - \hat{\mathbf{a}}_y \cos \theta] \exp[-jk(\tilde{x} \cos \theta + \tilde{y} \sin \theta)] \\ &= \frac{1}{\eta_0} [\hat{\mathbf{a}}_x \sin \theta - \hat{\mathbf{a}}_y \cos \theta] \exp[-jk((-x) \cos \theta + y \sin \theta)] \end{aligned} \quad (40b)$$

By using (36), the fields $\mathbf{E}^a$ and $\mathbf{H}^a$ are obtained as

$$\mathbf{E}^a = \hat{\mathbf{a}}_z \exp[-jk((-x) \cos \theta + y \sin \theta)] \quad (41a)$$

$$\mathbf{H}^a = -\frac{1}{\eta_0} (\hat{\mathbf{a}}_x \sin \theta + \hat{\mathbf{a}}_y \cos \theta) \exp[-jk((-x) \cos \theta + y \sin \theta)] \quad (41b)$$

Defining the propagation vector as $\mathbf{k}_2 = k(-\cos \theta \hat{\mathbf{a}}_x + \sin \theta \hat{\mathbf{a}}_y)$, these fields can be expressed as

$$\mathbf{E}^a = \hat{\mathbf{a}}_z \exp(-j\mathbf{k}_2 \cdot \mathbf{r}) \quad (42a)$$

$$\mathbf{H}^a = \frac{1}{\omega \mu_0} (\mathbf{E}^a \times \mathbf{k}_2) \quad (42b)$$

We can conclude from the field expressions that the perfectly matched Veselago medium yields negative refraction and the left-handed triplet $(\mathbf{E}^a, \mathbf{H}^a, \mathbf{k}_2)$, implying that the wave vector $\mathbf{k}_2$ and Poynting's vector $\mathbf{P}$ are anti-parallel.

So far, we assumed that stretching is applied in x coordinate only. This approach can be generalized by defining the mapping in all three coordinate variables. We define the coordinate transformation as follows:

$$\tilde{x} = \begin{cases} x & x < 0 \\ \int_0^x s_x(x') dx' & x > 0 \end{cases} \quad (43a)$$

$$\tilde{y} = \begin{cases} y & y < 0 \\ \int_0^y s_y(y') dy' & y > 0 \end{cases} \quad (43b)$$

$$\tilde{z} = \begin{cases} z & z < 0 \\ \int_0^z s_z(z') dz' & z > 0 \end{cases} \quad (43c)$$

where $s_x = s_y = s_z = -\alpha - j\sigma$. Defining a new set of fields $\mathbf{E}^a = (-\alpha - j\sigma)\tilde{\mathbf{E}}$ and $\mathbf{H}^a = (-\alpha - j\sigma)\tilde{\mathbf{H}}$, this set of fields satisfy Maxwell's equations in an anisotropic medium with $\bar{\bar{\mu}} = \mu_0 \bar{\bar{\Lambda}}$ and $\bar{\bar{\epsilon}} = \epsilon_0 \bar{\bar{\Lambda}}$, where

$$\bar{\bar{\Lambda}} = \begin{bmatrix} s_y s_z / s_x & 0 & 0 \\ 0 & s_z s_x / s_y & 0 \\ 0 & 0 & s_x s_y / s_z \end{bmatrix} \quad (44)$$

Letting $\alpha = 1$ and $\sigma \rightarrow 0$, we obtain $\bar{\bar{\Lambda}} = \pm \bar{\bar{\mathbf{I}}}$ depending on the octant where the point (x, y, z) is situated.

Consider the 2D case where $s_z = 1$, as shown in Fig. 5(b). Let the region $\Omega_1 = \{(x, y) \in \mathbb{R}^2 | x < 0, y < 0\}$ be free space and assume a field variation in the form of a plane wave given by

$$\mathbf{E} = \hat{\mathbf{a}}_z \exp(-j\mathbf{k}_1 \cdot \mathbf{r}) \quad (45a)$$

$$\mathbf{H} = \frac{1}{\omega \mu_0} (\mathbf{k}_1 \times \mathbf{E}) \quad (45b)$$

where $\mathbf{k}_1 = k(\hat{\mathbf{a}}_x \cos \theta + \hat{\mathbf{a}}_y \sin \theta)$. After extending $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ to complex space and obtaining $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$, the fields in the DNG media are obtained via $(\mathbf{E}^a, \mathbf{H}^a)$ —identified as $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ — as follows:

(i) In the region $\Omega_2 = \{(x, y) \in \mathbb{R}^2 | x > 0, y < 0\}$, $s_x = -1$ and $s_y = 1$. Then,

$$\bar{\bar{\Lambda}} = -\bar{\bar{\mathbf{I}}} \quad (46a)$$

$$\mathbf{E} = \hat{\mathbf{a}}_z \exp(-j\mathbf{k}_2 \cdot \mathbf{r}) \quad (46b)$$

$$\mathbf{H} = \frac{1}{\omega \mu_0} (\mathbf{E} \times \mathbf{k}_2) \quad (46c)$$

where $\mathbf{k}_2 = k(-\cos \theta \hat{\mathbf{a}}_x + \sin \theta \hat{\mathbf{a}}_y)$.

(ii) In the region $\Omega_3 = \{(x, y) \in \mathbb{R}^2 | x < 0, y > 0\}$, $s_x = 1$ and $s_y = -1$. Then,

$$\bar{\bar{\Lambda}} = -\bar{\bar{\mathbf{I}}} \quad (47a)$$

$$\mathbf{E} = \hat{\mathbf{a}}_z \exp(-j\mathbf{k}_3 \cdot \mathbf{r}) \quad (47b)$$

$$\mathbf{H} = \frac{1}{\omega \mu_0} (\mathbf{E} \times \mathbf{k}_3) \quad (47c)$$

where $\mathbf{k}_3 = k(\hat{\mathbf{a}}_x \cos \theta - \hat{\mathbf{a}}_y \sin \theta)$.

(iii) In the region $\Omega_4 = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$, $s_x = -1$ and $s_y = -1$. Then,

$$\bar{\bar{\Lambda}} = \bar{\bar{\mathbf{I}}} \quad (48a)$$

$$\mathbf{E} = \hat{\mathbf{a}}_z \exp(-j\mathbf{k}_4 \cdot \mathbf{r}) \quad (48b)$$

$$\mathbf{H} = \frac{1}{\omega \mu_0} (\mathbf{k}_4 \times \mathbf{E}) \quad (48c)$$

where $\mathbf{k}_4 = k(-\hat{\mathbf{a}}_x \cos \theta - \hat{\mathbf{a}}_y \sin \theta)$.

The expressions (45-48) can be interpreted as follows:

(i) The constitutive parameters in the regions Ω_1 and Ω_4 are ϵ_0 and μ_0 . The region Ω_1 is originally free space, and the region Ω_4 corresponds to the overlapping of the two perfectly matched DNG layers. It is remarkable that (44) implies

(a) If two perfectly matched DNG layers overlap, the medium is equivalent to free space (2D corner region).

(b) If three perfectly matched DNG layers overlap, the medium is again DNG (3D corner region).

(ii) In region Ω_4 , $\mathbf{k}_4 = -\mathbf{k}_1$.

(iii) If we choose a closed surface S enclosing the corner, the net power flow through the surface is zero, as can be concluded from the Poynting's vectors in the four quadrants.

As a special application of the approach introduced in this section, we can construct a perfectly matched anisotropic Veselago slab. Let us consider a Veselago medium occupying the region $\Omega_s = \{(x, y, z) \in \mathbb{R}^3 | x_1 < x < x_2\}$, and the region outside the slab is free space. The complex coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ can be expressed as

$$\tilde{x} = \begin{cases} x & x < x_1 \\ x_1 + s_x(x - x_1) & x_1 < x < x_2, \\ x_1 + s_x(x_2 - x_1) + (x - x_2) & x > x_2 \end{cases} \quad (49)$$

$$\tilde{y} = y, \quad \tilde{z} = z$$

The stretching variable s_x is chosen as $s_x = -\alpha - j\sigma$, where $\alpha > 0$. In the limit as $\sigma \rightarrow 0$, (49) becomes

$$\tilde{x} = \begin{cases} x & x < x_1 \\ x_1 - \alpha(x - x_1) & x_1 < x < x_2 \\ x_1 - \alpha(x_2 - x_1) + (x - x_2) & x > x_2 \end{cases} \quad (50)$$

Note that when $\alpha = 1$, the slab becomes perfectly matched isotropic Veselago slab with $\epsilon = -\epsilon_0$ and $\mu = -\mu_0$.

The slab is perfectly matched to free space, and acts as a perfect lens, with more general focusing properties, under certain conditions. To illustrate the lens-like action, let us consider the 2D TM_z case, where a line-source, extending to infinity along the z -direction, is represented by the current density as follows:

$$\mathbf{J} = \hat{\mathbf{a}}_z I \delta(\mathbf{r} - \mathbf{r}_s) \quad (51)$$

where I is a constant (amp/m). The source is located at $\mathbf{r}_s = (x_s, y_s)$, where $x_s < x_1$. Starting with Maxwell's equations, the scalar Helmholtz equation for the z -component of the radiated electric field (E_z) can be written as

$$\nabla^2 E_z + k^2 E_z = j\omega\mu_0 I \delta(\mathbf{r} - \mathbf{r}_s) \quad (52)$$

The analytical solution can be computed by

$$E_z(x, y) = -\frac{k\eta}{4} I H_0^{(2)}(k|\mathbf{r} - \mathbf{r}_s|) \quad (53)$$

where $H_0^{(2)}$ is the Hankel function of the second kind of zero-th order, η is the intrinsic impedance of the medium,

and $|\mathbf{r} - \mathbf{r}_s| = \sqrt{(x - x_s)^2 + (y - y_s)^2}$. The mapped version of $E_z(x, y)$ to complex space is $E_z(\tilde{x}, y)$, and using (50) the solution in the presence of the perfectly matched Veselago slab becomes

$$E_z(x, y) = -\frac{k\eta}{4} I \begin{cases} H_0^{(2)}\left(k\sqrt{(x - x_s)^2 + (y - y_s)^2}\right) & x < x_1 \\ H_0^{(2)}\left(k\sqrt{(x_1 - \alpha(x - x_1) - x_s)^2 + (y - y_s)^2}\right) & x_1 < x < x_2 \\ H_0^{(2)}\left(k\sqrt{(x_1 - \alpha(x_2 - x_1) + (x - x_2) - x_s)^2 + (y - y_s)^2}\right) & x > x_2 \end{cases} \quad (54)$$

It is clear from (54) that a focus within the slab (i.e., $x_1 < x_{fs} < x_2$) will occur at

$$x_{fs} = \left(1 + \frac{1}{\alpha}\right)x_1 - \frac{1}{\alpha}x_s \quad (55)$$

provided that the following inequalities are satisfied

$$x_s < x_1 \quad (56)$$

$$x_1 - x_s < \alpha(x_2 - x_1) \quad (57)$$

The inequality (56) always holds since the source is outside the slab, and (57) indicates that the distance between the source and the slab must be less than the slab width ($d = x_2 - x_1$) multiplied by α . Under these conditions, the position of the focus outside the slab will be

$$x_{fo} = x_s + (1 + \alpha)d \quad (58)$$

We can conclude that the distance from the dipole to the focus outside can be adjusted by changing the parameter α in the anisotropic Veselago slab. In other words, the anisotropic slab rather than the isotropic Veselago slab provides flexibility in the lens design. To demonstrate the concepts in the design of anisotropic Veselago slab, we plot the contours representing the real part of the electric field in Fig. 6 for different levels of anisotropy, namely for various values of α . The parameters of the slab can be easily visualized on the graphs.

IV. CONCLUSIONS

We reviewed complex coordinate approaches used in electromagnetics. A complex coordinate transformation is based on extending the real coordinate space to a complex coordinate space. We focused on two complex space PML methods (locally-conformal and multi-center PML methods) which are used for mesh truncation. We concluded that these methods are implemented by just replacing the real coordinates by complex coordinates inside the PML, and they provide great flexibility to design conformal PMLs over arbitrary geometries. Afterwards, we discussed the design of perfectly matched double negative (DNG) layer using complex coordinates, and illustrated the focusing feature of the resulting metamaterials. The proposed designs were validated through some numerical simulations performed by the finite element method.

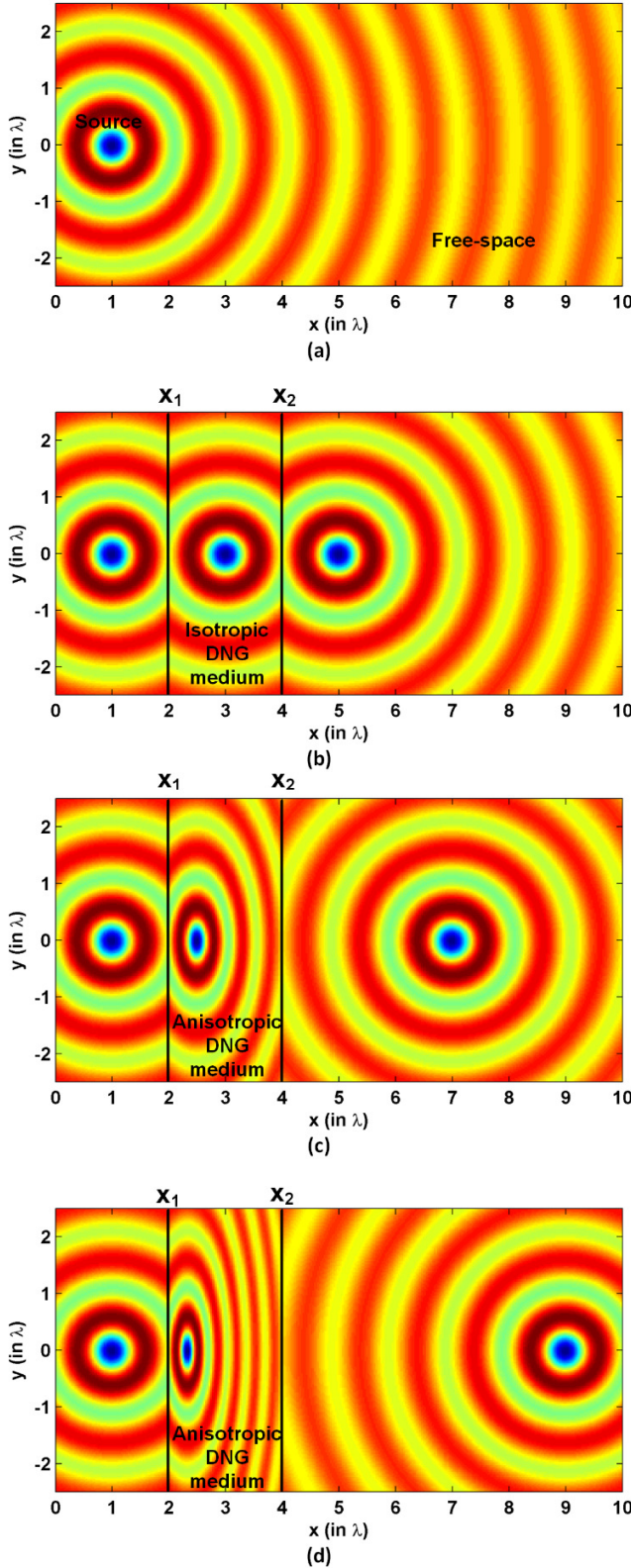


Fig. 6. Perfectly matched Veselago slab with $d = 2\lambda$: (a) Line-source at $x_s = 1\lambda$ in free-space ($\alpha = -1$); (b) Isotropic DNG medium with $\alpha = 1$. Focus is at $x_{fs} = 3\lambda$ inside, $x_{fo} = 5\lambda$ outside; (c) Anisotropic DNG medium with $\alpha = 2$. Focus is at $x_{fs} = 2.5\lambda$ inside, $x_{fo} = 7\lambda$ outside; (d) Anisotropic DNG medium with $\alpha = 3$. Focus is at $x_{fs} = 2.33\lambda$ inside, $x_{fo} = 9\lambda$ outside. ($\lambda = 1\text{m}$)

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